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EIGENVALUE ASYMPTOTICS AND A NON-LINEAR SCHRÖDINGER EQUATION

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ABSTRACT

The equation

$$(*) \qquad -\Delta u + qu + f(x,u) = \lambda u, \quad u \in W^{1,2}(\mathbb{R}^N)$$

is considered, where q is bounded below and $q(x) \to \infty$ as $|x| \to \infty$. Under appropriate conditions on the perturbation term $f(x,u)$ it is shown that given any $r > 0$, (*) has an infinite sequence $(\lambda_{n,r})_{n\in\mathbb{N}}$ of eigenvalues, each $\lambda_{n,r}$ being associated with an eigenfunction $u_{n,r}$ which satisfies $\int_{\mathbb{R}^N} |u_{n,r}|^2 = r^2$. Information about the behaviour of $\lambda_{n,r}$ for large n is provided. The proofs rely on the compactness of the embedding of a certain weighted Sobolev space in an L^p space; this is proved in §2.

1. Introduction

Let q be a smooth real-valued function on $\mathbb{R}^N$ $(N \geq 3)$ such that $q(x) \to \infty$ as $|x| \to \infty$, and suppose that q is bounded below by some positive number. Then it is well-known that the problem

$$(\mathrm{L}) \qquad -\Delta u + qu = \lambda u, \quad u \in W^{1,2}(\mathbb{R}^N)$$

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has a sequence (λ_n^0) of positive eigenvalues, each repeated according to multiplicity, with $\lambda_n^0 \le \lambda_{n+1}^0$ for all $n \in \mathbb{N}$ and $\lambda_n^0 \to \infty$ as $n \to \infty$. These may be characterised, by the classical Courant principle, as minimax values of the quadratic functional ϕ_0, where

$$\phi_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} \{|\nabla u(x)|^2 + q(x)|u(x)|^2\} dx,$$

on the surface $\{u : \int_{\mathbb{R}^N} |u(x)|^2 dx = \text{constant}\}$. Note that ϕ_0 is well-defined for precisely those u belonging to the weighted Sobolev space

$$W_q^{1,2}(\mathbb{R}^N) := \left\{ u : \int_{\mathbb{R}^N} (|\nabla u|^2 + qu^2) dx < \infty \right\},$$

which is plainly contained in $W^{1,2}(\mathbb{R}^N)$; the discreteness of the spectrum of $-\Delta + q$ is equivalent to the compactness of the embedding of $W_q^{1,2}(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ (see [5], Chap. VIII). Under appropriate additional conditions on q the asymptotic behaviour of the counting function

$$N_0(\lambda) := \sum_{\lambda_n^0 \le \lambda} 1$$

can be determined: it then turns out (see [5], Chap. XI) that as $\lambda \to \infty$,

$$\text{(A)} \qquad N_0(\lambda) \sim \omega_N (2\pi)^{-N} \int_{\mathbb{R}^N} (\lambda - q(x))_+^{N/2} dx$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$ and $g(\lambda) \sim h(\lambda)$ means that $g(\lambda)/h(\lambda) \to 1$ as $\lambda \to \infty$. A particular case in which formula (A) holds occurs (see [5], Chap XI) when there is a constant $a > 1$ and positive constants a_1, a_2, a_3 such that for all $x, y \in \mathbb{R}^N$,

$$a_1(|x|^a - 1) \le q(x) \le a_2(|x|^a + 1)$$

and

$$|q(x) - q(y)| \le a_3 |x - y| (\max\{|x|, |y|\})^{a-1}.$$

In the present paper we consider non-linear perturbations of problem (L) of the form

$$\text{(NL)} \qquad -\Delta u + qu + f(x, u) = \lambda u, \quad u \in W^{1,2}(\mathbb{R}^N),$$

where it is supposed that

$$\text{(H)} \qquad \begin{cases} f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \quad \text{is continuous and such that for some} \\ a > 0 \quad \text{and } b \in L^2(\mathbb{R}^N), \\ |f(x,s)| \leq a|s|^{p-1} + b(x), \quad 2 \leq p \leq 2N/(N-2) \end{cases}$$

for all $x \in \mathbb{R}^N$ and all $s \in \mathbb{R}$. Under this assumption, the functional

$$\phi(u) = \int_{\mathbb{R}^N} \left\{ \frac{1}{2}|\nabla u(x)|^2 + \frac{1}{2}q(x)|u(x)|^2 + F(x, u(x)) \right\} dx,$$

where $F(x,u) = \int_0^u f(x,s)ds$, is well-defined and of class C^1 on $W_q^{1,2}(\mathbb{R}^N)$, and (weak) solutions of (NL) can be sought as critical points of ϕ under the constraint $\int_{\mathbb{R}^N} |u(x)|^2 dx = \text{constant}$. Now if $W_q^{1,2}(\mathbb{R}^N)$ is compactly embedded in $L^p(\mathbb{R}^N)$, where p is as in (H), then ϕ will satisfy the Palais-Smale condition, so that if in addition,

$$\text{(O)} \qquad f(x,-s) = -f(x,s) \qquad \text{for all} \quad (x,s) \in \mathbb{R}^N \times \mathbb{R},$$

then a standard application of the Liusternik-Schnirel'mann (LS) minimax theory ([2],[8]) will establish the existence of infinitely many eigenvalue-eigenfuction pairs solving (NL).

Thus as in the linear case, the existence of countably many eigenvalues of (NL) hinges on the fact that a certain embedding is compact. In §2 we use a simple interpolation argument to prove that if $1 \leq t < N$ and $t \leq p < t^*$, where t^* is the Sobolev conjugate of t, then $W_q^{1,t}(\mathbb{R}^N)$ is compactly embedded in $L^p(\mathbb{R}^N)$. In the subsequent sections we use a similar interpolation inequality, giving a bound for the L^p norm of u in terms of its L^2 and $W_q^{1,2}$ norms, first to verify the requirements of the LS theory and then to estimate the eigenvalues of (NL). Our final result is as follows:

THEOREM: *Assume that f satisfies (O) and (H) with $2 \leq p < 2 + 4N^{-1}$, and suppose also that $f(x,s)s \geq 0$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$. Then for each $r > 0$, (NL) has an infinite sequence $(\lambda_{n,r})_{n \in \mathbb{N}}$ of eigenvalues, each $\lambda_{n,r}$ being associated with an eigenfunction $u_{n,r}$ satisfying $\int_{\mathbb{R}^N} |u_{n,r}(x)|^2 dx = r^2$, such that*

$$\lambda_{n,r} = \lambda_n^0 + O((\lambda_n^0)^\alpha),$$

where $\alpha = (p-2)N/4$.

Under suitable extra conditions on q it then follows that as $\lambda \to \infty$,

$$N_r(\lambda) := \sum_{\lambda_{n,r} \leq \lambda} 1 \sim \omega_N (2\pi)^{-N} \int_{\mathbb{R}^N} (\lambda - q(x))_+^{N/2} dx.$$

For work on the asymptotic distribution of eigenvalues of quasilinear elliptic operators on *bounded* subsets of $\mathbb{R}^N$ we refer to Chiappinelli ([3],[4]) and Moscatelli and Thompson [7].

2. Interpolation inequalities and compactness

We recall that when $t \in [1,\infty)$ the norm on the Sobolev space $W^{1,t}(\mathbb{R}^N)$ is $|\cdot|_{1,t}$, where

$$|u|_{1,t}^t = \int_{\mathbb{R}^N} \{|\nabla u(x)|^t + |u(x)|^t\}dx;$$

given any measurable subset Ω of $\mathbb{R}^N$ and any $p \in [1,\infty]$ we shall denote the norm on $L^p(\Omega)$ by $|\cdot|_{p,\Omega}$, writing $|\cdot|_{p,\mathbb{R}^N} = |\cdot|_p$ for simplicity. For any $q : \mathbb{R}^N \to \mathbb{R}$ which is measurable and positive almost everywhere, the weighted Sobolov space $W_q^{1,t}(\mathbb{R}^N)$ is defined to be the space of all (equivalence classes of) real functions u with distributional derivatives $\frac{\partial u}{\partial x_i}$ $(i = 1,\ldots,N)$ such that

$$|u|_{1,t,q} := \left\{\int_{\mathbb{R}^N} (|\nabla u|^t + q|u|^t)dx\right\}^{1/t}$$

is finite. If there is a positive constant c such that $q(x) \geq c$ for all $x \in \mathbb{R}^N$, then $W_q^{1,t}(\mathbb{R}^N)$ is a Banach space when endowed with the norm $|\cdot|_{1,t,q}$; if $N > t$, it is continuously embedded in $L^p(\mathbb{R}^N)$ for all $p \in [t,t^*]$, where $t^* = Nt/(N-t)$ is the Sobolev conjugate of t. Indeed, for any such q, clearly

$$|u|_{1,t} \leq \text{const.}|u|_{1,t,q}, \tag{1}$$

that is, $W_q^{1,t}(\mathbb{R}^N)$ is continuously embedded in $W^{1,t}(\mathbb{R}^N)$. Moreover, we know that $W_q^{1,t}(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$ for any $p \in [t,t^*]$ (see [5], Theorem V.3.7); that is,

$$|u|_p \leq \text{const.}|u|_{1,t}. \tag{2}$$

Our object here is to establish the *compactness* of the embedding of $W_q^{1,t}(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ under appropriate conditions. We begin with an interpolation inequality.

PROPOSITION 1: *Let $q : \mathbb{R}^N \to \mathbb{R}$ be measurable and such that for some $c > 0$, $q(x) \geq c$ for all $x \in \mathbb{R}^N$, let $t \in [1,N)$ and $p \in [t,t^*]$, and define $\beta = \beta(p)$ by*

$$\beta(p) = t(t^* - p)/(t^* - t). \tag{3}$$

Then given any $\gamma \in [0, \beta(p)]$, there exists $C > 0$ such that for all $u \in W_q^{1,t}(\mathbb{R}^N)$,

$$(4) \qquad |u|_p^p \le C|u|_t^\gamma |u|_{1,t,q}^{p-\gamma}.$$

Moreover, if $t \le p < t^*$, then there is a positive constant C such that for all measurable subsets Ω of $\mathbb{R}^N$ and all $u \in W_q^{1,t}(\mathbb{R}^N)$,

$$(5) \qquad |u|_{p,\Omega}^p \le CQ(\Omega)|u|_{1,t,q}^p.$$

where

$$Q(\Omega) = |1/q|_{\infty,\Omega}^{\beta/t}.$$

Proof: (i) If $\gamma = 0$, (4) follows directly from (1) and (2). If $0 < \gamma < t$, then by Hölder's inequality,

$$\int_{\mathbb{R}^N} |u(x)|^p dx = \int_{\mathbb{R}^N} |u(x)|^\gamma |u(x)|^{p-\gamma} dx$$
$$\le \left(\int_{\mathbb{R}^N} |u(x)|^t dx\right)^{\gamma/t} \left(\int_{\mathbb{R}^N} |u(x)|^{(p-\gamma)t/(t-\gamma)} dx\right)^{(t-\gamma)/t},$$

that is,

$$(6) \qquad |u|_p^p \le |u|_t^\gamma |u|_s^{p-\gamma}$$

where

$$(7) \qquad s = (p-\gamma)t/(t-\gamma).$$

Note that $s \ge t$ for any $\gamma \in (0, t)$, while $s \le t^*$ if, and only if, $\gamma \le \beta(p)$. It follows that if $p \in (t, t^*]$, then $\gamma \le \beta(p) < t$ and (4) results from (6) and (2). In the remaining case, in which $\gamma = \beta(p) = t = p$, (4) is trivial.

(ii) To prove (5) we argue in a similar fashion. For any measurable $\Omega \subset \mathbb{R}^N$ we have

$$\int_\Omega |u(x)|^p dx = \int_\Omega (q(x))^{-\beta/t} (q(x))^{\beta/t} |u(x)|^p dx$$
$$\le |1/q|_{\infty,\Omega}^{\beta/t} \int_\Omega (q(x))^{\beta/t} |u(x)|^p dx.$$

If $p = t$, then $\beta = t$ and the proof of (5) is thus accomplished. If $t < p < t^*$, put

$$\int_\Omega (q(x))^{\beta/t} |u(x)|^p dx = \int_\Omega \{(q(x)^{1/t}|u(x)|\}^\beta |u(x)|^{p-\beta} dx.$$

Thus the use of Hölder's inequality as above, with γ replaced by β, yields

$$\begin{aligned}\int_\Omega (q(x))^{\beta/t}|u(x)|^p dx &\le \left(\int_{\mathbb{R}^N} q(x)|u(x)|^t dx\right)^{\beta/t} |\, u|\,^{p-\beta}_{(p-\beta)t/(t-\beta)} \\ &\le C|\, u|\,^{\beta}_{1,t,q}|\, u|\,^{p-\beta}_{1,t,q} \\ &= C|\, u|\,^{p}_{1,t,q}.\end{aligned}$$

The inequality (5) is now immediate. ∎

THEOREM 2: *Let q be as in Proposition 1 and suppose that $q(x) \to \infty$ as $|x| \to \infty$; let $t \in [1, N)$ and $t \le p < t^*$. Then the embedding of $W_q^{1,t}(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ is compact.*

Proof: Let (u_n) be a bounded sequence in $W_q^{1,t}(\mathbb{R}^N)$, so that there exists $M > 0$ such that for all $n \in \mathbb{N}$,

$$|\, u_n|\,_{1,t,q} \le M \tag{8}$$

As in Proposition 1 let $Q(\Omega) = |\, 1/q|\,^{\beta/t}_{\infty,\Omega}$, and for each $R > 0$ set $Q_R = Q(\Omega_R)$, where $\Omega_R = \{x \in \mathbb{R}^N; |x| \ge R\}$. From (5) and (8) we have for all $m, n \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^N} |u_m(x) - u_n(x)|^p dx &= \int_{|x|<R} |u_m(x) - u_n(x)|^p dx + \int_{\Omega_R} |u_m(x) - u_n(x)|^p dx \\ &\le \int_{|x|<R} |u_m(x) - u_n(x)|^p dx + CQ_R(2M)^p.\end{aligned}$$

The assumptions on q imply that $Q_R \to 0$ as $R \to \infty$; thus given any $\varepsilon > 0$, there exists $R_0 > 0$ such that for all $m, n \in \mathbb{N}$ and all $R > R_0$,

$$\int_{\mathbb{R}^N} |u_m(x) - u_n(x)|^p dx \le \int_{|x|<R} |u_m(x) - u_n(x)|^p dx + \varepsilon.$$

Since $W^{1,t}(\Omega)$ is compactly embedded in $L^p(\Omega)$ when $\Omega = B(O, R)$, the open ball in $\mathbb{R}^N$ with centre O and radius R, we see that there is a subsequence $(u_{n(k)})$ of (u_n) which converges in $L^p(\mathbb{R}^N)$, and the result follows. ∎

3. Existence of eigenvalues

We say that (u, λ) is an eigenfunction–eigenvalue pair of (NL) if $u \in H := W_q^{1,2}(\mathbb{R}^N)$, $u \ne 0$, $\lambda \in \mathbb{R}$ and

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + quv + f(x, u)v) dx = \lambda \int_{\mathbb{R}^N} uvdx \tag{9}$$

for all $v \in H$; it is assumed that $N \geq 3$. The function f is subject to the condition

$$(\mathrm{H}) \quad \begin{cases} f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \quad \text{is continuous and such that for some } a > 0 \\ \text{and} \quad b \in L^2(\mathbb{R}^N), \\ |f(x,s)| \leq a|s|^{p-1} + b(x) \quad \text{for some} \quad p \in [2, 2N/(N-2)] \quad \text{and all} \\ (x,s) \in \mathbb{R}^N \times \mathbb{R}. \end{cases}$$

Setting $F(x,u) = \int_0^u f(x,s)ds$ and

$$\phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + q|u|^2)dx + \int_{\mathbb{R}^N} F(x, u(x))dx$$

$$(10) \qquad = \phi_0(u) + \int_{\mathbb{R}^N} f(x, u(x))dx,$$

$$(11) \qquad g(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx,$$

we see that (9) may be rewritten as

$$(12) \qquad \phi'(u)v = \lambda g'(u)v \quad \text{for all } v \in H,$$

where $\phi'(u), g'(u)$ denote the derivatives of ϕ, g at the point u. Thus to find eigenfunctions of (NL) with given $L^2(\mathbb{R}^N)$ norm, for $\int_{\mathbb{R}^N} u^2 dx = r^2$, say, consists in finding critical points of ϕ on the manifold

$$(13) \qquad M_r = \left\{ u \in H : g(u) = \frac{1}{2}r^2 \right\},$$

the corresponding eigenvalues appearing as Lagrange multipliers. Note that if (u, λ) is such a solution pair, then $\lambda = r^{-2}\phi'(u)u$, as follows from (12) on taking $v = u$.

THEOREM 3: *Let q be as in Theorem 2; assume that (H) and (O) hold and that either*

$$(\mathrm{H1}) \qquad (\mathrm{H}) \text{ holds with } \quad 2 \leq p < 2 + 4N^{-1}$$

or

$$(\mathrm{H2}) \qquad f(x,s)s \geq 0 \quad \text{for all} \quad (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

Then given any $r > 0$, there are denumerably many eigenfunction-eigenvalue pairs $(u_{n,r}, \lambda_{n,r})$ of (NL) with $\int_{\mathbb{R}^N} u_{n,r}^2 dx = r^2$ for all $n \in \mathbb{N}$, while as $n \to \infty$,

$$(14) \qquad \int_{\mathbb{R}^N} (|\nabla u_{n,r}|^2 + q u_{n,r}^2)dx \to \infty, \quad \lambda_{n,r} \to \infty.$$

Proof: For each $r > 0$ and $n \in \mathbb{N}$ set

$$K_n(r) = \{K \subset M_r : K \text{ compact and symmetric, } \gamma(K) \geq n\} \tag{15}$$

(where $\gamma(K)$ denotes the *genus* of K [8], corresponding to the original *category* of Liusternik and Schnirel'mann) and

$$c_n(r) = \inf_{K_n(r)} \sup_K 2\phi(u). \tag{16}$$

By the Liusternik–Schnirel'mann principle (see, for example, Theorem 20 in Browder [2]), it will be enough to show that ϕ is bounded below on M_r and satisfies the Palais–Smale condition on M_r to ensure that the $c_n(r)$ are attained and are critical levels of ϕ on M_r; that is, there exist $u_{n,r} \in M_r$, $\lambda_{n,r} \in \mathbb{R}$ such that

$$2\phi(u_{n,r}) = c_n(r) \tag{17}$$

and

$$\phi'(u_{n,r}) = \lambda_{n,r} g'(u_{n,r}). \tag{18}$$

We give below the proof when f satisfies (H1); the case in which it satisfies (H2) can be dealt with in a similar manner but is simpler as then $F(x,s) \geq 0$ and thus $\phi(u) \geq \phi_0(u)$.

(i) To prove that ϕ is bounded below on M_r, for each $r > 0$, we first note that (H) implies that

$$|F(x,s)| \leq c|s|^p + b(x)|s|$$

so that, by the Schwarz inequality,

$$\int_{\mathbb{R}^N} |F(x,u(x))|dx \leq c \int_{\mathbb{R}^N} |u(x)|^p dx + d\left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right)^{1/2} \tag{19}$$

where $d = |\, b|_2$. Similarly

$$\int_{\mathbb{R}^N} |f(x,u(x))u(x)|dx \leq c \int_{\mathbb{R}^N} |u(x)|^p dx + d\left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right)^{1/2}. \tag{20}$$

Next we use inequality (4) with $\gamma = \beta$; on setting $2\alpha = (p - \beta) = (p-2)N/2$, this becomes

$$|\, u|_p^p \leq C|\, u|_2^\beta \,|\, u|_{1,2,q}^{2\alpha} \tag{21}$$

and we conclude that on M_r,

$$\int_{\mathbb{R}^N} |F(x,u(x))|dx \le cr^\beta |u|_{1,2,q}^{2\alpha} + rd. \tag{22}$$

This implies that, in particular,

$$\begin{aligned} 2\phi(u) &= 2\phi_0(u) + 2\int_{\mathbb{R}^N} F(x,u(x))dx \\ &\ge |u|_{1,2,q}^2 - cr^\beta |u|_{1,2,q}^{2\alpha} - rd \quad (u \in M_r). \end{aligned} \tag{23}$$

The desired result now follows since the assumption $p < 2 + 4N^{-1}$ is equivalent to $\alpha < 1$. In fact this shows that ϕ is *coercive on* M_r, that is, $\phi(u) \to \infty$ if $|u|_{1,2,q} \to \infty$, $u \in M_r$.

(ii) (The Palais–Smale condition). We have to show that any sequence (u_n) in M_r along which $\phi(u_n)$ is bounded and $\phi_r'(u_n) \to 0$ contains a convergent subsequence; here ϕ_r' denotes the derivative of ϕ along M_r, namely

$$\phi_r'(u) = \phi'(u) - r^{-2}(\phi'(u)u)g'(u).$$

Define operators A, B, C in H by the rules that for all $u, v \in H$,

$$\begin{aligned} (Au,v) &= \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + quv + f(x,u)v)dx \\ &= (u,v) + (Cu,v), \\ (Bu,v) &= \int_{\mathbb{R}^N} uvdx, \end{aligned}$$

where $(\cdot,\cdot)$ stands for the inner product in H. Thus

$$\phi'(u)v = (Au,v), \quad g'(u)v = (Bu,v)$$

for all $u, v \in H$. Hence

$$\phi_r'(u)v = (Au,v) - r^{-2}(Au,u)(Bu,v)$$

and the condition $\phi_r'(u_n) \to 0$ in the dual H' of H is equivalent to

$$A_r(u_n) := Au_n - r^{-2}(Au_n,u_n)Bu_n \to 0 \tag{24}$$

in H. The key fact in the proof that the Palais–Smale condition holds is that B and C are *compact*. To show this, first note that if G denotes the Nemytskii operator induced by f, then C can be viewed as the composition

$$H \overset{i}{\hookrightarrow} L^p(\mathbb{R}^N) \overset{G}{\longrightarrow} L^{p'}(\mathbb{R}^N) \approx (L^p(\mathbb{R}^N))' \overset{i^*}{\hookrightarrow} H' \approx H$$

where $\approx$ denotes isometric isomorphism; thus essentially, $C = i^* \circ G \circ i$. By the growth assumption on f, $G : L^p(\mathbb{R}^N) \to L^{p'}(\mathbb{R}^N)$ is continuous and maps bounded sets onto bounded sets (see [6]); since i is compact, by Theorem 2, it follows that C is compact. The compactness of B is established in a similar manner.

Now let (u_n) be a sequence in M_r such that $(\phi(u_n))$ is bounded and $A_r(u_n) \to 0$. Then for some $c_0 > 0$ and all $n \in \mathbb{N}$, it follows from (23) that

$$c_0 \geq 2\phi(u_n) \geq |\, u_n|\,^2_{1,2,q} - cr^\beta |\, u_n|\,^{2\alpha}_{1,2,q} - rd,$$

and so $|\, u_n|_{\,1,2,q} \leq c_1$ for some c_1; that is, (u_n) is bounded. Since B is compact, there is a subsequence of (u_n), denoted again by (u_n) for convenience, such that (Bu_n) is convergent. Also, the real sequence with n^{th} term

$$(Au_n, u_n) = |\, u_n|\,^2_{1,2,q} + (Cu_n, u_n)$$

is bounded and we may assume, by passage to a subsequence, that it converges, too; thus by (24), (Au_n) converges. But $A = I + C$, where I is the identity map; as C is compact we may again suppose that (Cu_n) is convergent, and so (u_n) is convergent, as required.

(iii) We now prove the relations (14) concerning $u_{m,r}$ and $\lambda_{m,r}$ on the assumption that the critical levels $c_m(r) \to \infty$ as $m \to \infty$. For a proof that this assumption holds we refer the reader to [4] (a similar argument appears on page 365 of Krasnosel'skii's book [6]); the argument hinges upon the properties of the genus together with the facts that ϕ is coercive on M_r and M_r is weakly closed in H since H is compactly embedded in $L^2(\mathbb{R}^N)$.

First observe that, by (19) and (4) (with $\gamma = 0$)

$$2\phi(u) = 2\phi_0(u) + 2\int_{\mathbb{R}^N} F(x, u(x))dx \leq |\, u|\,^2_{1,2,q} + c|\, u|\,^p_{1,2,q} + d|\, u|_{\,1,2,q}.$$

Since $2\phi(u_{n,r}) = c_n(r) \to \infty$, it follows that $|\, u_{n,r}|_{\,1,2,q} \to \infty$. Moreover, by (18),

$$r^2\lambda_{n,r} = \phi'(u_{n,r})u_{n,r} = \int_{\mathbb{R}^N} \{|\nabla u_{n,r}|^2 + qu^2_{n,r} + f(x, u_{n,r})u_{n,r}\}dx. \tag{25}$$

From (20) we see that, just as in the proof of (22), we have

$$\int_{\mathbb{R}^N} |f(x,u)u|dx \le cr^\beta |u|_{1,2,q}^{2\alpha} + rd$$

for all $u \in M_r$. Thus

$$r^2\lambda_{n,r} \ge |u_{n,r}|_{1,2,q}^2 - cr^\beta |u_{n,r}|_{1,2,q}^{2\alpha} - rd.$$

As $|u_{n,r}|_{1,2,q} \to \infty$ and $\alpha < 1$, this implies that $\lambda_{n,r} \to \infty$ as $n \to \infty$. The proof of Theorem 3 is complete. ∎

4. Asymptotic behaviour of eigenvalues

Here we improve the result obtained in Theorem 3 that the 'non-linear' eigenvalues $\lambda_{n,r}$ tend to ∞ as $n \to \infty$ by comparing them with the eigenvalues of the linear problem (L). To do this we use the following

LEMMA 4: *Let q be as in Theorem 3 and let $\{\lambda_n^0 : n \in \mathbb{N}\}$ be the set of all eigenvalues of (L), arranged in increasing order and repeated according to multiplicities. Then for each $r > 0$,*

$$r^2\lambda_n^0 = \inf_{K_n(r)} \sup_K 2\phi_0(u) \tag{26}$$

where $K_n(r)$ is as in (15).

This is a reformulation, based on the properties of the genus, of the Courant minimax principle: for the proof, see [3] or (in a slightly different context) Section 6.7 of Berger's book [1].

THEOREM 5: *Let q be as in Theorem 3; assume that f satisfies (O) and both (H1) and (H2). For each $r > 0$, let $\lambda_{n,r}$ be the eigenvalues of (NL) given by Theorem 3. Then for any $r > 0$,*

$$\lambda_{n,r} = \lambda_n^0 + O((\lambda_n^0)^\alpha), \quad \alpha = (p-2)N/4 < 1.$$

Proof: By the estimate (22),

$$\begin{aligned} \phi(u) &= \phi_0(u) + \int_{\mathbb{R}^N} F(x,u(x))dx \\ &\le \phi_0(u) + cr^\beta(\phi_0(u))^\alpha + rd \end{aligned} \tag{27}$$

for all $u \in M_r$. If $g : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function, it is straightforward to check that for any $n \in \mathbb{N}$,

$$\inf_{K_n(r)} \sup_K g(2\phi_0(u)) = g\left(\inf_{K_n(r)} \sup_K 2\phi_0(u)\right),$$

so that, if $2\phi(u) \leq g(2\phi_0(u))$ on M_r, then Lemma 4 shows that

$$c_n(r) = \inf_{K_n(r)} \sup_K 2\phi(u) \leq \inf_{K_n(r)} \sup_K g(2\phi_0(u)) = g(r^2\lambda_n^0).$$

Using this with $g(t) = t + cr^\beta t^\alpha + rd$ we have, by (27),

$$c_n(r) \leq r^2\lambda_n^0 + cr^{\beta+2\alpha}(\lambda_n^0)^\alpha + rd.$$

Also, by the sign assumption (H2),

$$\phi(u) = \phi_0(u) + \int_{\mathbb{R}^N} F(x, u(x))dx \geq \phi_0(u),$$

and thus

$$c_n(r) \geq \inf_{K_n(r)} \sup_K 2\phi_0(u) = r^2\lambda_n^0.$$

We therefore have that

$$|c_n(r) - r^2\lambda_n^0| \leq cr^{\beta+2\alpha}(\lambda_n^0)^\alpha + rd. \tag{28}$$

On the other hand, (17) and (18) yield

$$c_n(r) - r^2\lambda_{n,r} = 2\phi(u_{n,r}) - \phi'(u_{n,r})u_{n,r}$$

and hence, writing for simplicity u_n instead of $u_{n,r}$,

$$\begin{aligned} c_n(r) - r^2\lambda_{n,r} &= \int_{\mathbb{R}^N} \{|\nabla u_n|^2 + qu_n^2 + 2F(x, u_n(x))\}dx \\ &\quad - \int_{\mathbb{R}^N} \{|\nabla u_n|^2 + qu_n^2 + f(x, u_n(x))u_n\}dx \\ &= \int_{\mathbb{R}^N} \{2F(x, u_n(x)) - f(x, u_n(x))u_n\}dx. \end{aligned}$$

Use of (22) and the similar bound for $\int f(x,u)udx$ thus gives

$$\begin{aligned} |c_n(r) - r^2\lambda_{n,r}| &\leq cr^\beta(\phi_0(u_n))^\alpha + rd \\ &\leq cr^\beta(\phi(u_n))^\alpha + rd \\ &\leq cr^\beta(c_n(r))^\alpha + rd. \qquad (29) \end{aligned}$$

Note that the above estimates hold under the 'general' growth condition (H). However, under the restriction $p \in [2, 2+4/N)$ imposed by (H1), then $\alpha = (p-2)N/4 < 1$ and (28) gives in particular

$$c_n(r) \sim r^2 \lambda_n^0$$

as $n \to \infty$, for each fixed $r > 0$. Together with (29) this yields

$$|c_n(r) - r^2 \lambda_{n,r}| \le c r^{\beta+2\alpha} (\lambda_n^0)^\alpha + rd,$$

and (28) and (29) now give

$$r^2 |\lambda_{n,r} - \lambda_n^0| \le c r^p (\lambda_n^0)^\alpha + rd$$

or

$$|\lambda_{n,r} - \lambda_n^0| \le c r^{p-2} (\lambda_n^0)^\alpha + r^{-1} d.$$

In other words,

$$\lambda_{n,r} = \lambda_n^0 + O((\lambda_n^0)^\alpha)$$

for each fixed $r > 0$; thus in particular $\lambda_{n,r} \sim \lambda_n^0$ as $n \to \infty$. The proof of the Theorem is complete. ∎

Note that if q satisfies appropriate additional conditions, then the asymptotic estimate (A) holds for $N_0(\lambda) = \sum_{\lambda_n^0 \le \lambda} 1$, and in fact $N_0(\lambda)$ is bounded above and below by constant multiples of λ^a for some $a > 0$: see [5], Chapter XI, §3. Theorem 5 then shows that if q satisfies these extra conditions, then for each $r > 0$,

$$N_r(\lambda) := \sum_{\lambda_{n,r} \le \lambda} 1 \sim \omega_N (2\pi)^{-N} \int_{\mathbb{R}^N} (\lambda - q(x))_+^{N/2} dx$$

as $\lambda \to \infty$.

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